Indian Statistical Institute, Bangalore
B.Math (Hons.) II Year, Second Semester
Semestral Examination
Optimization

Time: 3 hours   April 27, 2011   Instructor: Pl.Muthuramalingam
Maximum marks: 50

1. For the system

$$x_1 + 2x_2 - 3x_3 + x_4 = 7$$
$$2x_1 \qquad - 7x_3 + x_4 = 9,$$

$x_j \geq 0$ for all $j = 1, 2, 3, 4$ find all basic solutions, basic feasible solutions, nondegenerate bfs and degenerate bfs. [7]

Hint: Find a solution for $a_1x + b_1y = c_1, a_2x + b_2y = c_2$.

2. a) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k \in R^n$ and $S = \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_k\mathbf{v}_k : \lambda_i \geq 0, \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1\}$. Let $f : R^n \longrightarrow R$ be any linear map. Show that $\max_S f = \max_i f(\mathbf{v}_i)$. [2]

b) Let P be a polygonal region in R^2 given by

$$2x_1 + x_2 \geq 4$$
$$x_1 - x_2 \geq -4$$
$$-3x_1 + x_2 \geq -15$$
$$-x_1 \geq -7$$
$$x_1 \geq 0, x_2 \geq 0.$$

Let $c_1, c_2 \in R$. Define $g : R^2 \longrightarrow R$ by $g(x_1, x_2) = c_1x_1 + c_2x_2$. Determine $\max_P g$ and $\min_P g$ interms of c_1, c_2. [3]

3. Let $R_n^{++} = \{\mathbf{p} = (p_1, p_2, \cdots p_n)$ with $p_i > 0$ for each $i\}$. Let $\triangle_n = \{(x_1, x_2, \cdots x_n) : x_i \geq 0$ for each i and $\sum x_i = 1\}$. Define, for $\mathbf{p} \in R_n^{++}$, $S_\mathbf{p} : \triangle_n \longrightarrow \triangle_n$ by $S_\mathbf{p}(x_1, x_2, \cdots, x_n) = \frac{(p_1x_1, p_2x_2, \cdots p_nx_n)}{\sum p_jx_j}$

a) Let $\mathbf{p}, \mathbf{q} \in R_n^{++}$. Let $\mathbf{r} = (r_1, r_2, \cdots, r_n)$ with $r_j = p_jq_j$. Find a relation between $S_\mathbf{p} \circ S_\mathbf{q}$ and $S_\mathbf{r}$ and prove your claim. [3]

b) Show that $S_\mathbf{p}$ is $1-1$, onto for each $\mathbf{p}$ in R_n^{++}. [1]

c) Show that $S_\mathbf{p}$ maps any straight line in $\triangle_n$ to a straight line. [3]

4. Let $A : R^n_{col} \longrightarrow R^m_{col}$ be a linear and onto map. Show that $\sup_{\mathbf{x}\varepsilon\triangle_n} \inf_i \sum_j a_{ij}x_j = \inf_{\mathbf{y}\varepsilon\triangle_m} \sup_j \sum_i a_{ij}y_i$. [6]

   Hint: If you need, conversion table from primal to dual form, it is given below

TABLE

Primal	Dual
row $i \sum_j a_{ij}x_j = b_i$	y_i real
row$p \sum_j a_{pj}x_j \geq b_p$	$y_p \geq 0$
variable j x_j real	$\sum_i y_i a_{ij} = c_j$
var q $x_q \geq 0$	$\sum_i y_i a_{iq} \leq c_q$
min $\sum c_j x_j$	max $\sum y_i b_i$

5. Define $f, g : R^2 \longrightarrow R, L : R^2 \times R \longrightarrow R$ by

   $f(x, y), = 2x^3 - 3x^2$

   $g(x, y) = (3 - x)^3 - y^2$

   $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$.

   Let D = {(x,y) : g(x,y) = 0}

   a) Show that $\max_D f$ is attained at $(3, 0)$. [3]

   b) If (x, y, λ) is a critical point for L ie $\partial_x L, \partial_y L, \partial_\lambda L = 0$ at (x, y, λ), then show that $(x, y, \lambda) \in \{(0, \pm\sqrt{27}, 0), (1, \pm\sqrt{8}, 0)\}$. [2]

   c) Find the rank of g' at $(3, 0)$. [1]

6. Let $A : R^n_{col} \longrightarrow R^{m_0}_{col}$ be linear onto.

   Let $A = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \cdots, \mathbf{a}^{(m_0)}, \mathbf{a}^{(m_0+1)}, \cdots, \mathbf{a}^{(n)}]$. be the Column representation of A. Let $B = [\mathbf{a}^{(1)}, \cdots, \mathbf{a}^{(m_0)}]$, $N = [\mathbf{a}^{(m_0+1)}, \cdots \mathbf{a}^{(n)}]$. Let $\mathbf{c} \in R^n_{col}.\mathbf{c} = (c_1, c_2, \cdots, c_n)^t$, Define $\mathbf{c}_B, \mathbf{c}_N$ by $\mathbf{c}_B = (c_1, c_2, \cdots, c_{m_0})^t$, $\mathbf{c}_N = (c_{m_0+1}, \cdots, c_n)^t$. Define the reduced cost vector $\mathbf{r}$ depending on $\mathbf{c}, B$ as $\mathbf{r}_B = \mathbf{0}, \mathbf{r}_N = \mathbf{c}_N - (B^{-1}N)^t\mathbf{c}_B$. Let $\mathbf{v}$ be any B basic feasible solution and $\mathbf{y}$ any basic feasible solutions for $A\mathbf{x} = \mathbf{b}$.

   a) Show that $-\mathbf{c}^t\mathbf{v} + \mathbf{c}^t\mathbf{y} = \mathbf{r}^t_N\mathbf{y}_N$. [3]

   b) Assume further $B = m_o \times m_o$ identity matrix. Let $\mathbf{v}$ be B basic non degenerate solution such that $\mathbf{c}^t \cdot \mathbf{v} = inf\{\mathbf{c}^t \cdot \mathbf{y} : \mathbf{y} \geq 0, A\mathbf{y} = \mathbf{b}\}$.

   Show that $\mathbf{r} \geq \mathbf{0}$. [6]

7. Let D be a convex set in R^n and $f : D \longrightarrow R$ a concave and C^1 function. Show that $\mathbf{x}^*$ is a global maxima for f if and may if $f'(\mathbf{x}^*)\mathbf{y} \leq 0$ for all $\mathbf{y}$ pointing into D at $\mathbf{x}^*$ [5]

8. Let $A, \mathbf{b}, B$ be as in $Q_n 6(a)$. For $m_0 + 1 \leq j \leq n$, $1 \leq l \leq m_0$ let $B(j, l) = \{\mathbf{a}^{(j)}\} \cup B/\{\mathbf{a}^{(l)}\}$. For j as above, let $\mathbf{a}^{(j)} = \sum_{i=1}^{m_0} \gamma_i^{(j)} \mathbf{a}^{(i)}$. Let $\mathbf{x}$ be any B basic feasible solution.

   a) Find a necessary sufficient condition so that $B(j, l)$ is a basis. [2]

   b) Assume that $\{i : \gamma_i^{(j)} > 0\}$ is non empty. Let $\theta_0^{(j)} = min\{\frac{x_i}{\gamma_i^{(j)}} : \gamma_i^{(j)} > 0\} = \frac{x_l}{\gamma_l^{(j)}}$ for some l in $\{1, 2, \cdots, m_0\}$. Define $\mathbf{y}^{(j)} = (y_1^{(j)}, y_2^{(j)}, \cdots, y_n^{(j)})^t$ by $y_i^{(j)} = x_i - \theta_0^{(j)} \gamma_i^{(j)}$. Then $\mathbf{y}^{(j)}$ is a $B(j, l)$ basic feasible solution. [5]